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Representations of a braid group with transpose symmetry and the related link invariants

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Abstract. Under a weight conservation condition when q is not a root of unity, Yang–Baxter relations are solved without the assumption that the upper-left triangle of each sub-block of the S matrix of the braid group representation (BGR) vanishes. A general form of BGR associated with B_n, C_n and D_n is obtained and shown to construct link invariants. The many-to-one correspondence between BGRs and link invariants is given.

1. Introduction

Due to the importance of the Yang–Baxter equation (YBE) in the study of quantum groups, knot theory and integrable models in quantum field theory and statistical mechanics etc [1–8], the study of the solutions of the YBE becomes an interesting subject. There have been several approach to the solutions of the YBE in recent years [7–10]. One of them follows the strategy that to solve Yang–Baxter relations (briefly, YBR means parameter-independent YBE) under a weight conservation condition [11], then to Baxterize them [10]. When solving YBR, the upper-left triangle of each non-vanishing sub-block is usually assumed to be null. However this assumption is not required by Markov properties which are compatible with YBR [11].

In this paper, starting from the structure of the BGR under a weight conservation condition, we solve YBR without the assumption that the upper-left triangle of each non-vanishing sub-block is null. In the next section, a brief illustration of notation of the extended Kauffman diagram is given. In section 3 we give a strategy to write out all possible YBEs according to the diagram structure of the BGR and actions of the permutation group S_3. A sequence of solutions of YBRs associated with B_n, C_n and D_n are obtained. The results are connected with the so-called non-standard BGR through actions of a permutation group S_N. In section 4, we show that link polynomials can be defined from the BGR obtained. The correspondence relationship between BGRs and link polynomials are discussed.

2. On notation of the extended Kauffman diagram

In this section, we briefly recall and illustrate the notation of the extended Kauffman diagram [5]. One advantage of diagrammatic notation is that various YBRs can be

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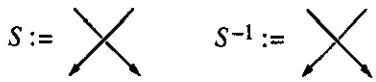
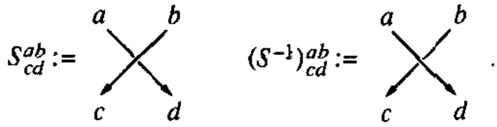
written down conveniently. Another advantage is that the polynomial for a given link is easy to calculate by expanding each crossing (intersection) as 'state' components multiplied with writhe factors.

In diagrammatic form, the S matrix of the BGR and its inverse are represented as

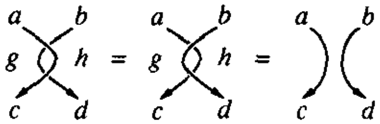


i.e.



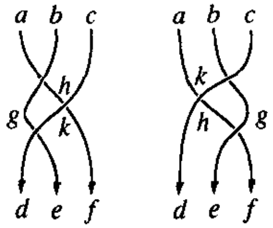
(2.1)

Then the unitary condition and YBR are depicted respectively as



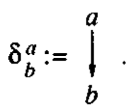
(2.2)

and

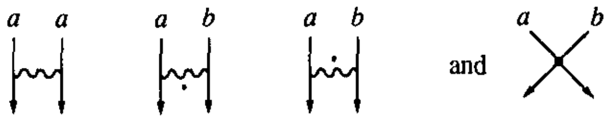


(2.3)

where an inner line connecting legs of two crossings implies the summation over the repeated labels on the legs, and a simple vertical arrow stands for a unit matrix i.e.



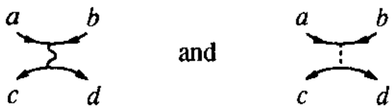
Furthermore, some special components of S matrix S^{aa}_{aa}, S^{ab}_{ab} $(a < b)$, S^{ab}_{ab} $(a > b)$ and S^{ab}_{ba} $(a \neq b)$ are denoted respectively by the following Kauffman 'state' diagram notation:



(2.4)

where the labels connected by the tip and tail of an arrow are supposed to be equal; if two full lines are connected by a wavy line, it is supposed that the sum of labels on one full line equals that on the other full line (in (2.4) a wavy line with a dot below stands for $a < b$ and that with a dot above for $a > b$).

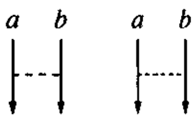
Obviously, other kinds of components of the S matrix need to be given diagrammatic notation. Referring to the principle contained in (2.4), a simple reasonable notation is



etc. (2.5)

The left one stands for S^{ab}_{cd} when $a \neq c$, $a \neq d$ but $a + b = c + d$ and the right one for the case when $a \neq c, a \neq d$ and $a + b \neq c + d$.

In order to simplify the discussion of YBR in the next section, it is not necessary to distinguish the first three situations of (2.4). We will use the following diagram notation:



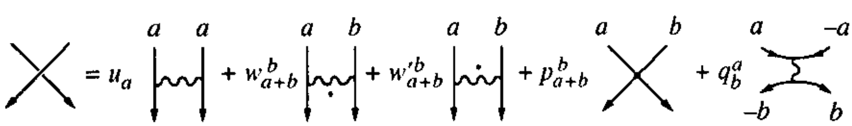
(2.6)

where the left includes $a = b$, $a < b$ and $a > b$, and the right includes $a < b$ and $a > b$ only.

One may notices that any matrix can be expressed as a linear combination of the 'states' in (2.4) and (2.5). However it is very difficult to solve YBR starting from a general matrix. The weight conservation condition provides a compatible method to eliminate many 'states' (i.e. restrict available components of the S matrix to be null) before solving YBR. The weight conservation condition when q is not a root of unity is the starting point of the following discussion.

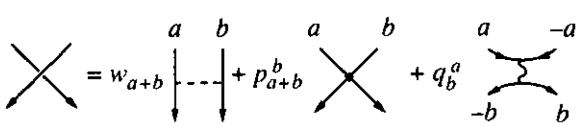
3. Braid group representations

In [11], we have given the structure of the BGR under the weight conservation condition for the cases of fundamental representations of B_n, C_n and D_n



(3.1)

where the sets of labels are $\ell = \{2n, 2n-2, \ldots, -2n+2, -2n\}$ for B_n; $\ell = \{2n-1, 2n-3, \ldots, -2n+3, -2n+1\}$ for C_n and D_n, $q^a_b = 0$ when $a \pm b = 0$, and $q^a_b = q^{-b}_{-a}$ and $p^a_c = p^{c-a}_c$ due to transpose symmetry. If the notation of (2.6) is adopted, (3.1) becomes



(3.2)

where $u_a := w^a_{2a}$, $\omega^b_{a+b} := w^b_{a+b}$ $(a < b)$ and $\omega'^b_{a+b} := w^b_{a+b}$ $(a > b)$. All the coefficients in (3.2) should be determined by YBR.

3.1. Yang–Baxter relations

Once the labels on the top and bottom of diagram (2.3) are given, one can write out a concrete equation (YBR). So a concrete YBR can be denoted symbolically by $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$. Because of the third term in (4.2), discussions of YBR in these cases are more complicated than in the case of A_n [12]. We need to consider not only $\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$ $(a+b \neq 0,\ b+c \neq 0)$ and those with a permutation of the bottom labels, but also $\begin{pmatrix} a & -a & b \\ a & -a & b \end{pmatrix}$, $\begin{pmatrix} b & -a & a \\ b & -a & a \end{pmatrix}$, $\begin{pmatrix} a & b & -b \\ a & -c & c \end{pmatrix}$, $\begin{pmatrix} -b & b & a \\ c & -c & a \end{pmatrix}$, $\begin{pmatrix} b & a & -b \\ -c & a & c \end{pmatrix}$ and those with a permutation of the bottom labels.

Due to the symmetry properties of (3.2), it is not necessary to depict diagram equations in all possible situations. Here YBR is invariant under interchange of the top and bottom labels. The YBR whose free labels are the image of others through a vertical mirror can be obtained from the original ones by

$$w_c^a \to w_c^{c-a} \qquad q_b^a \to q_{-b}^{-a} = q_a^b\,. \tag{3.4}$$

Using our procedure for writing YBR in the situations of $\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}$ $(a+b \neq 0,\ b+c \neq 0)$ associated with the elements of S_3, we obtain what we have obtained in the case of A_n (see [12]—here exclude $a+b=0$ and $b+c=0$). These equations have nothing to do with the coefficients q_b^a. The solutions are easily found:

$$\begin{aligned} &p_{a+b}^b = 1\ (a \neq b,\ a+b \neq 0) \qquad p_{2b}^b = 0 \\ &u_a := w_{2a}^a = \delta_a q^{\delta_a} \qquad (a \neq 0) \\ &w_{a+b}^b = (q-q^{-1})\chi_{a,b} \qquad (a \neq b,\ a+b \neq 0) \end{aligned} \tag{3.5}$$

where $\delta_a = \pm 1$ and $\chi_{a,b} = 1$ or 0 satisfying

$$\chi_{a,b} + \chi_{b,a} = 1 - \delta_{a,b}\,. \tag{3.6}$$

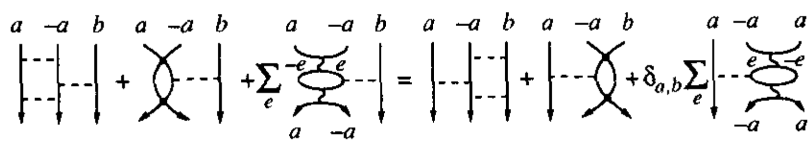
Meanwhile $\chi := (\chi_{a,b})$ are related to a primer matrix $\tilde{\chi}$ via any element of the permutation group S_N i.e.

$$\chi = M^t(\pi)\tilde{\chi}M(\pi) \qquad \pi \in S_N \tag{3.7}$$

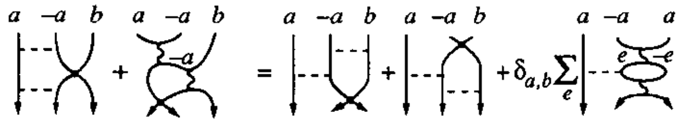
where $M(\pi) \in \mathrm{Mat}(S_N)$, $N = 2n+1$ for B_n, $N = 2n$ for C_n and D_n. The definition of the primer matrix is that $\tilde{\chi}_{a,b} = 1$ for $a < b$, $\tilde{\chi}_{a,b} = 0$ for $a \geqslant b$.

In the following we discuss the cases $\begin{pmatrix} a & -a & b \\ a & -a & b \end{pmatrix}$, $\begin{pmatrix} a & b & -b \\ a & -c & c \end{pmatrix}$ and $\begin{pmatrix} b & a & -b \\ -c & a & c \end{pmatrix}$. The other two cases can be written out directly by using (3.4). Letting S_3 act on $\begin{pmatrix} a & -a & b \\ a & -a & b \end{pmatrix}$ we have the following diagram equations:

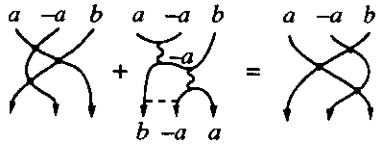
$\begin{pmatrix} a & -a & b \\ a & -a & a \end{pmatrix}$:



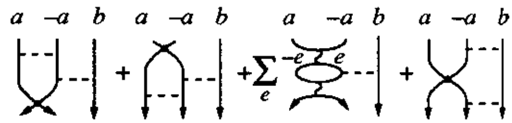
$\begin{pmatrix} a & -a & b \\ a & b & -a \end{pmatrix}$:



$\begin{pmatrix} a & -a & b \\ b & -a & a \end{pmatrix}$:

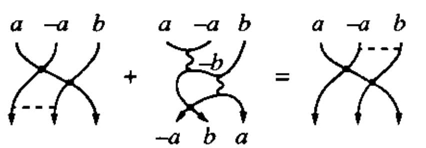


$\begin{pmatrix} a & -a & b \\ -a & a & b \end{pmatrix}$:



$\begin{pmatrix} a & -a & b \\ b & a & -a \end{pmatrix}$:

$\begin{pmatrix} a & -a & b \\ -a & b & a \end{pmatrix}$:



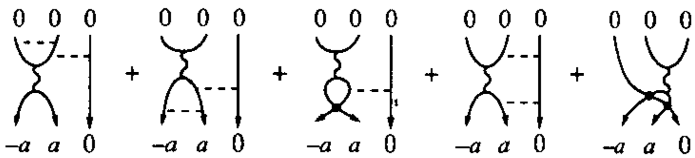
From the above diagram equations, we obtain

$$
\begin{aligned}
& w_0^{-a} w_{b-a}^b (w_0^{-a} - w_{b-a}^b) + w_{a+b}^b [(p_0^a)^2 - (p_{b-a}^b)^2] \\
& \qquad + \sum_e (q_e^a)^2 w_{e+b}^b - \delta_{a,b} \sum_e (q_a^e)^2 w_{a+e}^e = 0 \\
& p_{b-a}^b (w_{b-a}^b w_0^{-a} + w_{a+b}^b w_{b-a}^{-a} - w_0^{-a} w_{a+b}^b) - p_{a+b}^b q_{-b}^a q_b^a \\
& \qquad + \delta_{a,b} \sum_e q_{-a}^e q_a^e w_{a+e}^e = 0 \\
& w_{a+b}^a q_b^a q_a^b = 0 \\
& p_0^a (w_0^{-a} w_{b-a}^b + w_{a+b}^b w_0^a - w_{b-a}^b w_{a+b}^b) + \sum_e q_e^a q_a^{-e} w_{e+b}^b = 0 \\
& w_{a+b}^a q_b^a q_{-a}^b = 0 \\
& p_{b-a}^b q_{-b}^a q_a^{-b} = 0 \,.
\end{aligned}
\tag{3.8}
$$

Obviously $\begin{pmatrix} a & b & -b \\ a & -c & c \end{pmatrix}$ is equivalent to $\begin{pmatrix} a & b & -b \\ a & c & -c \end{pmatrix}$. As the latter is the result of $\pi_{23} \in S_3$ acting on the bottom labels of the former, we need only consider the situations associated with the right co-set representations, where S_3 is decomposed with respect to the right co-set of subgroup (id, π_{23}) i.e. $\begin{pmatrix} a & b & -b \\ a & -c & c \end{pmatrix}$, $\begin{pmatrix} a & b & -b \\ c & -c & a \end{pmatrix}$ and $\begin{pmatrix} a & b & -b \\ -c & a & c \end{pmatrix}$. Similarly, as $\begin{pmatrix} b & a & -b \\ -c & a & c \end{pmatrix}$ is equivalent to $\begin{pmatrix} b & a & -b \\ c & a & -c \end{pmatrix}$ which is $\pi_{13} \in S_3$ acting on the bottom labels of $\begin{pmatrix} b & a & -b \\ -c & a & c \end{pmatrix}$, we only need to consider $\begin{pmatrix} b & a & -b \\ -c & a & c \end{pmatrix}$, $\begin{pmatrix} b & a & -b \\ -c & c & a \end{pmatrix}$ and $\begin{pmatrix} b & a & -b \\ a & -c & c \end{pmatrix}$. These YBRs can be written out after depicting the related diagram equations carefully. For brevity we omit them here.

Because $0 \in \ell$ for B_n, there are several cases which are not contained in the above. They are $\begin{pmatrix} a & 0 & 0 \\ a & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \end{pmatrix}$, $\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ -a & a & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ -a & 0 & a \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & -a \end{pmatrix}$, which should be added to the case of B_n. For example

$\begin{pmatrix} 0 & 0 & 0 \\ -a & a & 0 \end{pmatrix}$:



tells us that

$$q^0_a[(w^0_0)^2 - p^{-a}_{-a}p^a_a] + q^0_a w^0_a(w^a_0 - w^0_0) + q^0_{-a}p^{-a}_0 w^0_{-a} = 0\,. \tag{3.9}$$

3.2. Braid group representations

Following the procedure in above discussion, we write out all YBRs, which involves a large number of equations. We find that the number of those equations are decreased dramatically after taking

$$q^a_b\, w^a_{b+a} = 0\,. \tag{3.10}$$

Then solutions which are compatible with (3.5) are found without much difficulty. The following are the results of the remainder coefficients beyond (3.5):

$$p^a_0 = \delta_a q^{-\delta_a},\ q^a_b = -\chi_{a,b}\,(q - q^{-1})\bar{q}^a_b$$
$$w^a_0 = \chi_{-a,a}(q - q^{-1})(1 - \bar{q}^{-a}_a) \qquad (a \neq 0)$$
$$\bar{q}^a_b := \begin{cases} (-1)^{\frac{a-b}{2}} u_a^{-1/2} u_b^{-1/2} \prod\limits_{e\in\ell} u_e^{-\chi_{a,e}\chi_{e,b}} & \text{for } B_n \\ \epsilon(ab) u_f^{-\theta(-ab)} u_a^{-1/2} u_b^{-1/2} \prod\limits_{e\in\ell} u_e^{-\chi_{a,e}\chi_{e,b}} & \text{for } C_n \\ u_f^{\theta(-ab)} u_a^{-1/2} u_b^{-1/2} \prod\limits_{e\in\ell} u_e^{-\chi_{a,e}\chi_{e,b}} & \text{for } D_n \end{cases} \tag{3.11}$$

where $u_0 = \delta_0$ for B_n, f is a fixed label such that $\chi_{-f,e}\chi_{e,f} = 0$ for any $e \in \ell$; $\theta(x)$ is the step function and $\epsilon(x)$ is the sign function. Since $p^a_0 = p^{-a}_0$ (transpose symmetry) and $p^a_0 = \delta_a q^{-a}$, then we must take

$$\delta_a = \delta_{-a}\,. \tag{3.12}$$

Likewise, $q^a_b = q^{-b}_{-a}$ requires that

$$\chi_{-a,-b} = \chi_{b,a} \qquad (a \pm b \neq 0)\,. \tag{3.13}$$

These relations do not appear in the case of A_n.

4. Link invariants

In this section we will show that link polynomials can be constructed from some of the above BGRs. We consider the BGRs in which $\chi = (\chi_{a,b})$ are transformed from the primer matrix $\tilde{\chi}$ via actions of a subgroup of S_N instead of S_N itself. This subgroup is supposed to keep relations (3.12) and (3.13) and denoted by $\mathcal{S}'$. Obviously, the elements $\pi \in S_N$ satisfying $\pi(-a) = -\pi(a)\ \forall a \in \ell - \{0\}$ guarantee relations (3.12) and (3.13) at least.

Link polynomials are defined by the following formula [13]:

$$P(A) = (\tau \cdot \bar{\tau})^{-m-\frac{1}{2}} \left(\frac{\bar{\tau}}{\tau}\right)^{e(A)/2} \phi(A) \qquad \phi(A) = \mathrm{str}(\mathrm{Hg}(A)) \tag{4.1}$$

where $\mathrm{str}(M) := \mathrm{tr}(\mathcal{H}M)$, $\mathcal{H} = \prod^m \otimes \eta$, $\eta^a_b = \delta_a \delta^a_b$ and $H = \prod^m \otimes h$; $g(A)$ stands for the matrix representation of $A \in B_m$. The definition of a diagonal matrix h is

$$h := M^t(\pi')\tilde{h}M(\pi') \qquad \pi' \in \mathcal{S} \subset S_N \tag{4.2}$$

and

$$\tilde{h} = \begin{cases} \delta^a_b\, q^{\mu - \delta_0 \epsilon(b) + \delta_b - 2\Sigma^{2n}_{c=b}\delta_c} & \text{for } B_n \\ \delta^a_b\, q^{\mu + \delta_1 \epsilon(b) + \delta_b - 2\Sigma^{2n-1}_{c=b}\delta_c} & \text{for } C_n \\ \delta^a_b\, q^{\mu - \delta_1 \epsilon(b) + \delta_b - 2\Sigma^{2n-1}_{c=b}\delta_c} & \text{for } D_n \end{cases} \tag{4.3}$$

where we have adopted the notation

$$\mu := \mathrm{tr}\eta = \sum_{b\in\ell} \delta_b\,. \tag{4.4}$$

The S matrices of BGRs for B_n, C_n and D_n have three distinct eigenvalues: $\lambda_1 = q$, $\lambda_2 = -q^{-1}$ for B_n, C_n and D_n but $\lambda_3 = q^{-\mu+1}$ for B_n, $\lambda_3 = -\delta_1 q^{-\mu-\delta_1}$ for C_n and $\lambda_3 = \delta_1 q^{-\mu+\delta_1}$ for D_n. Then cubic reduction relations of BGRs for these cases can be written out. After calculating τ and $\bar{\tau}$ which are determined by the requirements of Markov move II, i.e.

$$\sum_b \delta_b S^{ab}_{ab} h^b_b = \tau \qquad \sum_b \delta_b (S^{-1})^{ab}_{ab} h^b_b = \bar{\tau} \tag{4.5}$$

where τ and $\bar{\tau}$ are independent of a, we obtain from definition (4.1) and reduction relations of BGRs the following cubic skein relations:

$$\begin{aligned} B_n:\quad & q^{2(\mu-1)}P_{+2} - (q^{\mu} - q^{\mu-2} + 1)P_{+1} - (q^{-\mu} - q^{-\mu+2} + 1)P_0 \\ & + q^{-2(\mu-1)}P_{-1} = 0 \end{aligned} \tag{4.6}$$

$$\begin{aligned} C_n:\quad & q^{2(\mu+\delta_f)}P_{+2} - (q^{\mu+\delta_f+1} - q^{\mu+\delta_f-1} - \delta_f)P_{+1} \\ & + \delta_f(q^{-\mu-\delta_f-1} - q^{-\mu-\delta_f+1} - \delta_f)P_0 - \delta_f\ q^{-2(\mu+\delta_f)}P_{-1} = 0 \end{aligned} \tag{4.7}$$

$$\begin{aligned} D_n:\quad & q^{2(\mu-\delta_f)}P_{+2} - (q^{\mu-\delta_f+1} - q^{\mu-\delta_f-1} + \delta_f)P_{+1} \\ & - \delta_f(q^{-\mu+\delta_f-1} - q^{-\mu+\delta_f+1} + \delta_f)P_0 + \delta_f q^{-2(\mu-\delta_f)}P_{-1} = 0\,. \end{aligned} \tag{4.8}$$

It is known that the Jones polynomial and its two-variable extension (ie. HOMFLY polynomial) obey the quadratic skein relations, which are obtained via C^* algebra [14], and are shown to relate to the braid group representation of the A_n case [12]. The polynomials obtained above obey the cubic skein relation and are certainly the hierarchies of the Kauffman polynomial [5].

Let us observe the polynomials of an unknotted single loop (up to a scalar multiplier $(\tau \cdot \tilde{\tau})^{-m-\frac{1}{2}}$)

$$P(\bigcirc) = \mathrm{tr}(\eta M^t(\pi')\tilde{h}M(\pi')) = \mathrm{tr}(\tilde{\eta}\tilde{h}) . \tag{4.9}$$

From (4.3) and (4.9), we find that

$$B_n : \quad P(\bigcirc) = q^{\mu-2} + q^{\mu-4} + \cdots + q + 1 + q^{-1} + \cdots + q^{-\mu+4} + q^{-\mu+2} \tag{4.10}$$

$$C_n : \quad P(\bigcirc) = \begin{cases} q^{\mu} + q^{\mu-2} + \cdots + q^2 + q^{-2} + \cdots + q^{-\mu+2} + q^{-\mu} & (\delta_f = 1) \\ q^{\mu-2} + q^{\mu-4} + \cdots + q^2 + 2 + q^{-2} + \cdots + q^{-\mu+2} & (\delta_f = -1) \end{cases} \tag{4.11}$$

$$D_n : \quad P(\bigcirc) = \begin{cases} q^{\mu-2} + q^{\mu-4} + \cdots + q^2 + 2 + q^{-2} + \cdots + q^{-\mu+4} + q^{-\mu+2} \\ \qquad\qquad (\delta_f = 1) \\ q^{\mu} + q^{\mu-2} + \cdots + q^2 + q^{-2} + \cdots + q^{-\mu+2} + q^{-\mu} \\ \qquad\qquad (\delta_f = -1) . \end{cases} \tag{4.12}$$

From the cubic skein relation and the polynomial of a single loop, all the concrete polynomials for the link are determined. According to the results of skein relations and polynomials of a single loop we conclude that (i) B hierarchy: a variety of BGRs related to Lie algebra B_n having the same μ ($\mu < n$) and the so-called standard BGR related to $B_{(\mu-1)/2}$ define the same polynomials. (ii) C_n and D_n hierarchy: BGRs related to C_n with $\delta_f = 1$ ($\delta_f = -1$) and those related to D_n with $\delta_f = -1$ ($\delta_f = 1$) having the same μ define the same polynomials as those defined by the so-called standard BGR related to $C_{\mu/2}$ ($D_{\mu/2}$). So the correspondence between BGRs and polynomials is many-to-one. The BGRs having the same polynomial may have very different sizes.

5. Conclusions and discussions

In the previous sections, we solved YBRs under the simplification of weight conservation condition and transpose symmetry without the assumption of upper-left triangle of each non-vanishing sub-block being null. The weight conservation condition and transpose symmetry are respectively requirements of the Markov move I for a diagonal h and a sufficient condition for a third type of move [10]. This simplification lets us conveniently solve YBRs in terms of the Kauffman diagrammatic technique. Whether the BGR obtained in such a way can define link invariants depends only on whether the property of the Markov move II can be guaranteed.

We can show that the BGRs obtained in the previous section give rise to representations of the Birman–Wenzl algebra. Then these BGRs can be Baxterized to be solutions of parameter-dependent Yang–Baxter equation, i.e. $R(x)$ matrices.

Finally, we would like to make two remarks. If the transpose symmetry is given up, one will obtain multiparameter solutions of YBRs. If we consider that q is a root of unity, the weight conservation condition proposed in our previous paper will become a wider condition, related to cyclic representations of the braid group.

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